Failure properties of loaded fiber bundles having a lower cutoff in fiber threshold distribution

Srutarshi Pradhan* and Alex Hansen†
Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway
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The presence of lower cutoff in fiber threshold distribution may affect the failure properties of a bundle of fibers subjected to external load. We investigate this possibility—both in an equal load sharing (ELS) model and in a local load sharing (LLS) one using analytic as well as numerical methods. In the ELS model, the critical strength gets modified and, beyond a certain lower cutoff level, the whole bundle fails instantly (brittle failure) after the first fiber ruptures. Although the dynamic exponents for the order parameter, susceptibility, and relaxation time remain unchanged, the avalanche size distribution shows a gradual deviation from the mean field power law. A similar "instant failure" situation occurs in the LLS model at a lower cutoff level, which reduces to that of the equivalent ELS model at higher (high enough) dimensions. Also, the system size variation of the bundle's strength and the avalanche statistics show strong dependence on the lower cutoff level.

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I. INTRODUCTION

Critical behavior of the fracture-failure phenomena in disordered materials has attracted wide interest these days [1]. Among several model studies, the fiber bundle models (FBMs) capture almost correctly the collective static and dynamics of fracture failure in loaded materials. The two different versions of FBMs have been studied much. The equal load sharing (ELS) model [2,3] considers democratic (equal) sharing of applied load on the bundle, whereas in the local load sharing (LLS) model [4,5], only the nearest neighbors support the terminal load (stress) of a failed fiber. Experimentally it has been observed [6–8] that disordered material under increasing load shows well-defined power laws in terms of acoustic emissions prior to the global rupture. Such a power law in burst avalanches has been achieved analytically (and verified through simulations) by Hemmer and Hansen [9,10] in the ELS model. It has also been known for several decades that the static ELS model has a critical point [11–15], i.e., at a critical strength (σ_c) the bundle shows a phase transition from a state of partial failure (for $\sigma \leq \sigma_c$) to the state of total failure (for $\sigma > \sigma_c$). This failure dynamics has been solved analytically [16,17], which explores the critical behavior through the power law variation of the order parameter, susceptibility, and relaxation time. Also the mean-field universality of the ELS model has been established recently [18]. The extensive studies on the LLS model [4,5] suggest that the strength goes to "zero" value as the bundle size approaches infinity [19–21], thus excluding the possibility of any critical behavior. Another important observation is that no universal power law asymptotics exist for the avalanche statistics [10] in the LLS model. Attempts have also been made to study both the ELS and LLS models in a single framework introducing adjustable load sharing parameter [22–24] and a crossover from mean-field (ELS) to short-range (LLS) behavior has been reported.

*Electronic address: pradhan.srutarshi@phys.ntnu.no
†Electronic address: alex.hansen@phys.ntnu.no

So far in the FBM studies, people mainly considered different fiber threshold distributions starting from zero threshold. However, in reality every element (fiber) should have a finite (nonzero) strength threshold due to the cohesive force among the constituting molecules. Therefore the idea of a lower cutoff in fiber threshold distribution would be most welcome. Andersen *et al.* [13] considered first a lower cutoff in fiber threshold distribution and established the "tricritical behavior" in the mean-field (ELS) fiber bundle model. Such distributions with lower cutoff have also been considered to study the nonlinear response in the ELS mode [17] and to establish the universal behavior [18] of ELS failure dynamics. The exclusion of weaker fibers not only enhances the ultimate strength of the bundle, it can affect the failure properties of the bundle. To investigate such possibilities, in the present work we consider ELS and LLS models and proceed through analytic as well as numerical methods.

We organize this report as follows: After this brief introduction we study the effect of lower cutoff in the ELS model (Sec. II) and in the LLS model (Sec. III). The importance of such study and the physical significance of the observed results are discussed in the conclusion (Sec. IV). In the Appendix we apply our analytic formulations in two different situations of fiber threshold distribution.

II. ELS MODEL

A. Solutions of the recursive dynamics for equal load increment

We consider a fiber bundle model having N parallel fibers subjected to an external load or stress (load per fiber). The threshold strength of each fiber is determined by the stress value (σ_{th}) it can bear, and beyond which it fails. We consider fiber threshold distribution to have a lower cutoff (σ_L), i.e., a randomly distributed normalized density $\rho(\sigma_{th})$ has been chosen within the interval σ_L and 1 such that

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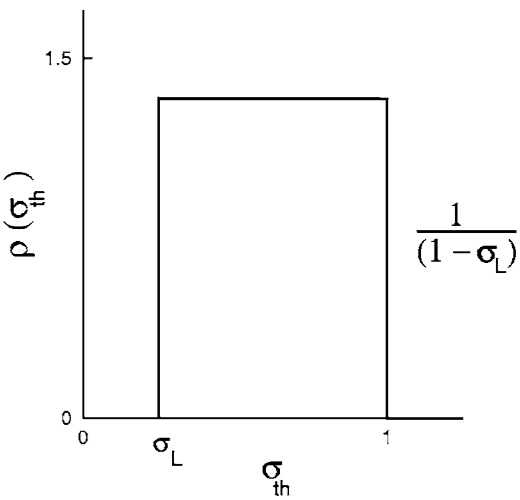

FIG. 1. The uniform density of fiber strength having a lower cutoff (σ_L).

$$\int_{\sigma_L}^{1} \rho(\sigma_{th}) d\sigma_{th} = 1. \tag{1}$$

We follow stepwise equal load increment [16,17] until the total failure of the bundle. The breaking dynamics starts when an initial stress σ ($>\sigma_L$) is applied on the bundle. Fibers having strength less than σ fail instantly, reducing the number of intact fibers and these fibers have to bear the applied load (ELS rule). Hence the effective stress (on intact fibers) increases and this compels some more fibers to break. These two sequential operations, the stress redistribution and further breaking of fibers, continue until an equilibrium is reached, where either the surviving fibers are strong enough to bear the applied load or all fibers fail.

The breaking dynamics can be represented by a recursion relation [16–18] in discrete time steps:

$$U_{t+1} = 1 - P(\sigma/U_t), \quad U_0 = 1; \tag{2}$$

where U_t is the fraction of total fibers that survive after time step t and $P(\sigma_t)$ is the cumulative distribution of corresponding density $\rho(\sigma_{th})$,

$$P(\sigma_t) = \int_{\sigma_L}^{\sigma_t} \rho(\sigma_{th}) d\sigma_{th}. \tag{3}$$

The time step indicates the number of stress redistributions at a fixed applied load.

At the equilibrium or steady state we get $U_{t+1} = U_t \equiv U^*$. This is a fixed point of the recursive dynamics and Eq. (2) can be solved at the fixed point for some particular strength distribution.

We choose the uniform density of fiber strength threshold having a lower cutoff (Fig. 1) to solve the recursive failure dynamics of the ELS model. Thus $\rho(\sigma_{th})$ has the form

$$\rho(\sigma_{th}) = \frac{1}{1-\sigma_L}, \quad \sigma_L < \sigma_{th} \leq 1. \tag{4}$$

The cumulative distribution becomes

$$P(\sigma_t) = \int_{\sigma_L}^{\sigma_t} \rho(\sigma_{th}) d\sigma_{th} = \frac{(\sigma_t - \sigma_L)}{(1-\sigma_L)}. \tag{5}$$

Therefore U_t follows a simple recursion relation:

$$U_{t+1} = \frac{1}{1-\sigma_L}\left(1 - \frac{\sigma}{U_t}\right), \tag{6}$$

which has fixed points [17,18]

$$U^*(\sigma) = \frac{1}{2(1-\sigma_L)}\left[1 \pm \left(1 - \frac{\sigma}{\sigma_c}\right)^{1/2}\right], \tag{7}$$

where

$$\sigma_c = \frac{1}{4(1-\sigma_L)}. \tag{8}$$

Beyond this critical strength (σ_c) the whole bundle fails instantly. The solution with (+) sign is the stable one, whereas the one with (−) sign gives unstable solution [17,18].

It is obvious that the critical strength (σ_c) cannot be less than σ_L. *As σ_c is a critical point (see Refs. [16–18]), there should be some critical exponents associated to σ_c. If none of the fibers fail ($\sigma < \sigma_L$), we cannot define order parameter, susceptibility, relaxation time, etc., which show critical behavior of the failure dynamics. Therefore, at σ_c the bundle should be in a partially broken (stable) state.* Putting the condition $\sigma_c \geq \sigma_L$ in Eq. (8), we get the upper bound of the lower cutoff: $\sigma_L < 1/2$. We can verify that for $\sigma_L > 1/2$ the recursion [Eq. (6)] does not give a stable fixed point except $U^*=0$. Also, putting $\sigma_L > 1/2$ in fixed point solution [Eq. (7)], we get $U^* > 1$, which is unrealistic. Therefore the critical strength of the ELS model is bounded by an upper limit: $\sigma_c \leq 1/2$. We present graphical solutions (Fig. 2) of the recursion relation [Eq. (6)] for $\sigma_L < 1/2$ (a) and $\sigma_L > 1/2$ (b). Clearly, we cannot get a fixed point ($U_{t+1} = U_t$) in (b).

From the solution [Eq. (7)] we can obtain the order parameter (O), susceptibility (χ), and relaxation time (τ) [16–18] of the failure process:

$$O = U^*(\sigma) - U^*(\sigma_c) \sim (\sigma_c - \sigma)^{-\alpha}, \quad \alpha = \frac{1}{2} \tag{9}$$

$$\chi = \left|\frac{dU^*(\sigma)}{d\sigma}\right| \sim (\sigma_c - \sigma)^{-\beta}, \quad \beta = \frac{1}{2} \tag{10}$$

$$\tau \sim (\sigma_c - \sigma)^{-\theta}, \quad \theta = \frac{1}{2}. \tag{11}$$

Therefore the variation of order parameter, susceptibility, and relaxation time remain unaffected by the presence of lower cutoff.

B. The instant failure situation in weakest fiber breaking approach

Now, we follow the weakest fiber breaking approach [3,9]: The applied load is tuned in such a way that only the weakest fiber (among the intact fibers) will fail after each

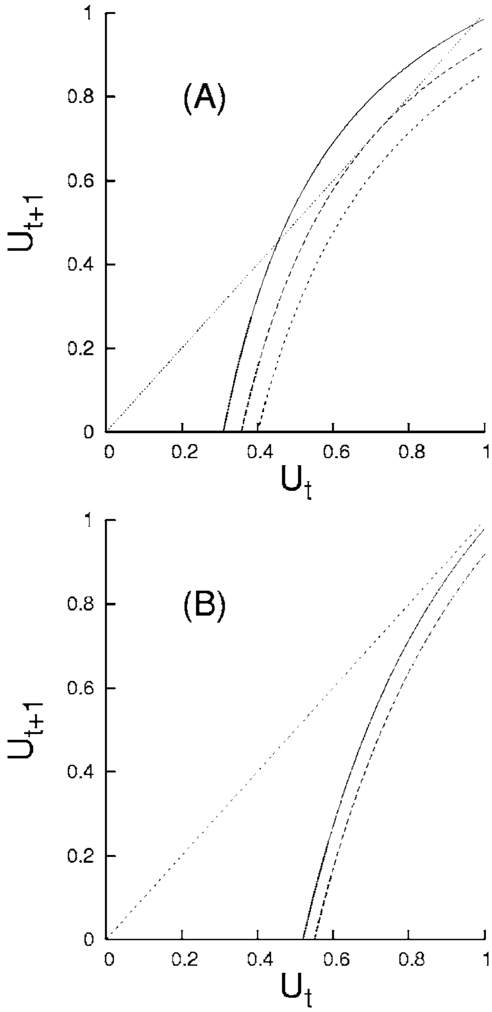

FIG. 2. The graphical solutions of Eq. (6): Straight lines represent the fixed points. In (a) $\sigma_L=0.3, \sigma_c=0.357$; solid curve, $\sigma<\sigma_c$; dashed line, $\sigma=\sigma_c$, dotted line, $\sigma>\sigma_c$. In (b) $\sigma_L=0.51$; solid curve, $\sigma=0.52$; dashed line, $\sigma=0.55$.

step of loading. We first find out the extreme condition when the whole bundle fails instantly after the first fiber ruptures. As the strength thresholds of N fibers are uniformly distributed within σ_L and 1, the weakest fiber fails at a stress σ_L (for large N). After this single fiber failure, the load will be redistributed within intact fibers resulting in a global stress $\sigma_f=N\sigma_L/(N-1)$. Now, the number of intact fibers having strength threshold below σ_f is

$$NP(\sigma_f) = N\int_{\sigma_L}^{\sigma_f} \rho(\sigma_{th})d\sigma_{th} = \frac{N(\sigma_f - \sigma_L)}{(1-\sigma_L)}. \tag{12}$$

The stress redistribution can break at least another fiber if $NP(\sigma_f)\geqslant 1$ and this "second" failure will trigger another failure and so on. Thus the successive breaking of fibers cannot be stopped until the complete collapse of the bundle. Clearly, there cannot be any fixed point (critical point) for such "instant failure" situation. Putting the value of σ_f we get

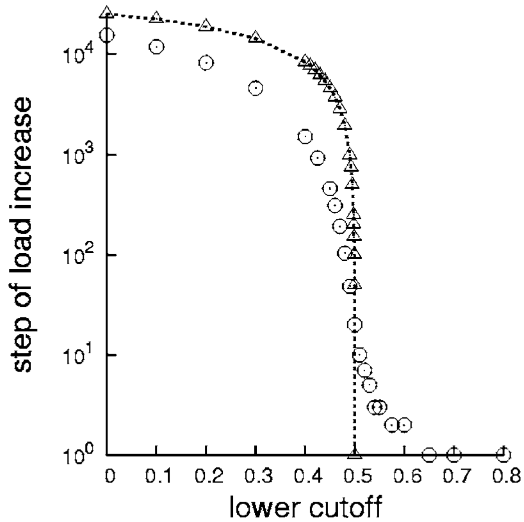

FIG. 3. The total step of load increase (until final failure) is plotted against σ_L for an ELS model having 50 000 fibers. The dotted line represents the analytic form [Eq. (17)], triangles are the simulated data for a strictly uniform strength distribution, and the circles represent the data (averages are taken for 5000 samples) for a uniform on average distribution.

$$\frac{N\left(\frac{N\sigma_L}{N-1} - \sigma_L\right)}{(1-\sigma_L)} \geqslant 1, \tag{13}$$

which gives

$$\sigma_L \geqslant \frac{(N-1)}{(2N-1)}. \tag{14}$$

For large N limit, the above condition can be written as $\sigma_L \geqslant 1/2$. Therefore, the condition to get a fixed point in the failure process is $\sigma_L < 1/2$.

We can also calculate how many steps are required to attain the final catastrophic failure for $\sigma_L < 1/2$. Let us assume that we have to increase the external load x times before the final failure. At each step of such load increment only one fiber fails. Then after x steps the following condition should be fulfilled to have a catastrophic failure:

$$N\int_{\sigma_i}^{\sigma_i[1+1/(N-x)]} \rho(\sigma_{th})d\sigma_{th} \geqslant 1, \tag{15}$$

where

$$\sigma_i = \sigma_L + \frac{x(1-\sigma_L)}{N}. \tag{16}$$

The solution gives

$$x = \frac{N}{2}\left(1 - \frac{\sigma_L}{1-\sigma_L}\right). \tag{17}$$

The above equation suggests that at $\sigma_L=1/2$, $x=0$. But in reality we have to put the external load once to break the weakest fiber of the bundle. Therefore, $x=1$ for $\sigma_L \geqslant 1/2$ (Fig. 3). To check the validity of the above calculation we

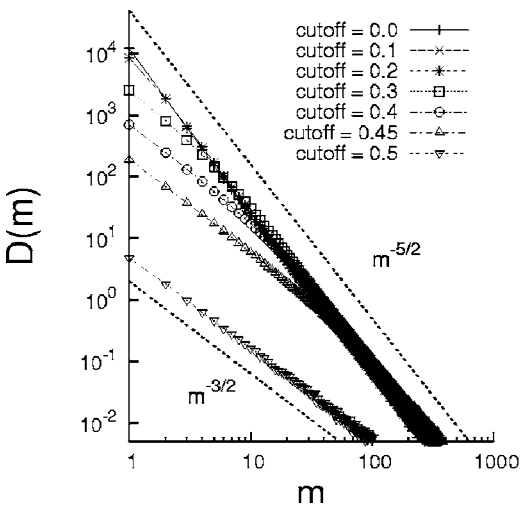

FIG. 4. The avalanche size distributions for different values of σ_L: $N=50\,000$ and averages are done over 10 000 sample. We have drawn two power laws (dotted lines) as reference lines to compare our numerical results.

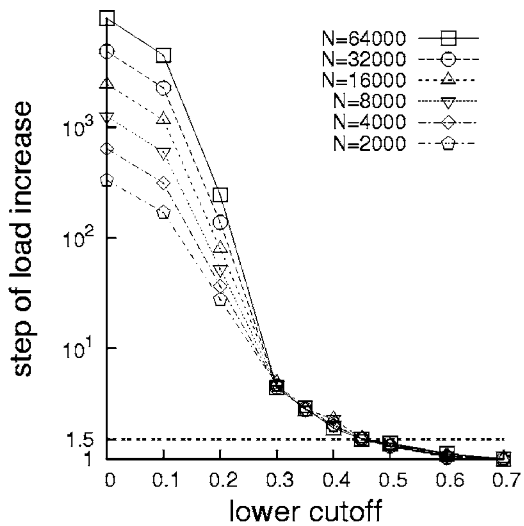

FIG. 5. Numerical estimate of the upper bound of σ_L in the LLS model: For $\sigma_L \geqslant 0.5$ the average step values go below 1.5, i.e., the bundle fails at one step in most of the realizations.

take "strictly uniform" and uniform on average distributions of fiber strength. In our strictly uniform distribution the strength of the kth fiber (among N fibers) is $\sigma_L+(1-\sigma_L)k/N$. We can see in Fig. 3 that the strictly uniform distribution exactly obeys the analytic formula (17) but the uniform on average distribution shows slight disagreement which comes from the fluctuation in the distribution function for finite system size. This fluctuation will disappear at the limit $N\rightarrow\infty$ where we expect perfect agreement.

C. Avalanche size distribution

During the failure process "avalanches" of different size appear where simultaneous failure of a number of fibers is termed as an "avalanche." To investigate whether the avalanche size distributions depend on the lower cutoff or not, we go for a numerical study. The result (Fig. 4) demands that for small avalanche sizes the distributions show a gradual deviation (depends on σ_L) from the mean-field result, although the big avalanches still follow the mean-field power law (exponent value $-5/2$) as analytically derived by Hemmer and Hansen [9]. We have checked this result for several system sizes and the above feature remains invariant. We should mention that according to Eqs. (37)–(39) of Ref. [9], the analytic treatment is valid for $\sigma_L<1/2$ as $1/2$ is the maximum [upper limit of the integration in Eq. (37)] of the stress-strain curve. The presence of σ_L cuts out the lower part of the stress-strain curve where the small avalanches are most likely to happen. Therefore, smaller avalanches get reduced in number with the increase of σ_L (Fig. 4). At the limit $\sigma_L\rightarrow 1/2$, the avalanche distributions seem to follow a new power law with exponent $-3/2$, which can be explained as follows: The fluctuation in threshold distribution gives rise to a scenario like the unbiased random walk of the bundle's strength around the maximum $1/2$, which in turn results in exponent $-3/2$ in avalanche distribution [12].

III. LLS MODEL

A. The instant failure situation

Now we consider the LLS model with uniform fiber threshold distribution having a lower cutoff σ_L (Fig. 1). We shall present a probabilistic argument to determine the upper limit of σ_L, beyond which the whole bundle fails at once. Following the weakest fiber breaking approach, the first fiber fails at an applied stress σ_L (for large N). As we are using periodic boundary conditions, the n nearest neighbors (n is the coordination number) bear the terminal stress of the failing fiber and their stress value rises to $\sigma_f=\sigma_L(1+1/n)$. Now, the number of nearest neighbors (intact) having strength threshold below σ_f is $(nn)_{fail}=nP(\sigma_f)$ [see Eq. (12)]. Putting the value of $P(\sigma_f)$ and σ_f we finally get

$$(nn)_{fail}=\frac{(\sigma_L)}{(1-\sigma_L)}. \quad (18)$$

If $(nn)_{fail}\geqslant 1$, then at least another fiber fails and this is likely to trigger a cascade of failure events resulting in complete collapse of the bundle. Therefore, to avoid the instant failure situation we must have $(nn)_{fail}<1$, from which we get the upper bound of σ_L:

$$\sigma_L<\frac{1}{2}. \quad (19)$$

As the above condition does not depend on the coordination number n, at any dimension the whole bundle is likely to collapse at once for $\sigma_L\geqslant 1/2$. It should be mentioned that the LLS model should behave almost like the ELS model at the limit of infinite dimensions and therefore the identical bound (of σ_L) in both the cases is not surprising (see the Appendix). We numerically confirm (Fig. 5) the above analytic argument [Eq. (19)] in one dimension. When the average step value goes below the 1.5 line, one step failure is the dominating mode then. We can find out the extreme limit of σ_L when all

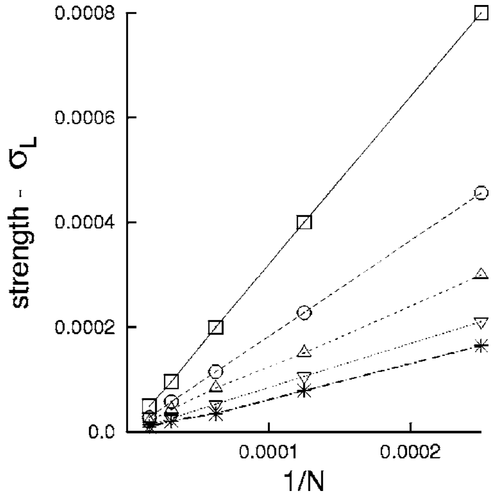

FIG. 6. The (strength -σ_L) is plotted against $1/N$ for different σ_L values: 0.3 (square), 0.35 (circle), 0.4 (up triangle), 0.45 (down triangle), and 0.5 (star). All the straight lines approach 0 value as $N \to \infty$.

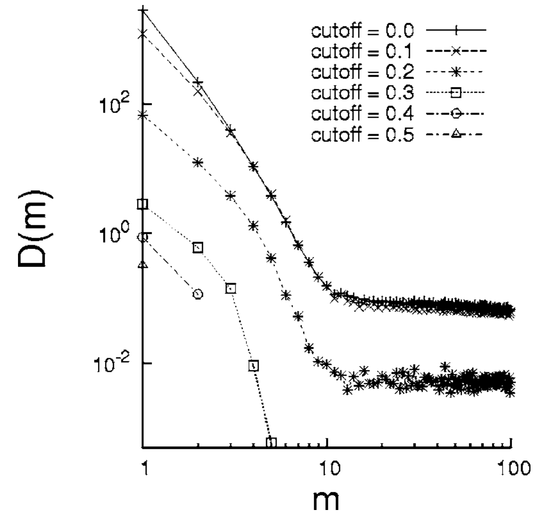

FIG. 7. Avalanche size distribution for several σ_L's in the LLS model: $N=20\,000$, averages are taken over 10 000 samples. The number of different avalanches decreases with the increase of σ_L value.

the nearest neighbors fail after the weakest fiber breaks. Then the LLS bundle collapses instantly for sure. Setting $(nn)_{fail} = n$ we get the condition $\sigma_L \geq n/(1+n)$, where the stress level of all the nearest neighbors crosses the upper cutoff 1 of the strength distribution. Clearly such failure is very rapid (like a chain reaction) and does not depend on the shape of the strength distributions, except for the upper cutoff. Also as n increases (ELS limit), σ_L for instant failure assumes the trivial value 1. Similar sudden failure in the FBM has been discussed by Moreno *et al.* [25] in the context of a "one-sided load transfer" model, which is different from the true LLS model we consider here.

B. Strength of the bundle

The local load sharing (LLS) scheme introduces stress enhancement around the failed fiber, which accelerates damage evolution. Therefore, a few isolated cracks can drive the system toward complete failure through growth and coalescence. The LLS model shows zero strength (for fiber threshold distributions starting from zero value) at the limit $N \to \infty$, following a logarithmic dependence on the system size (N) [19–21]. Recently Mahesh *et al.* [26] have proposed a probabilistic method of finding the asymptotic strength of bundles in the LLS mode. Now for threshold distributions having a lower cutoff (σ_L), the ultimate strength of the bundle cannot be less than σ_L. For such a uniform distribution (Fig. 1), we perform numerical simulations to investigate the system size variation of the bundle's strength. We observe (Fig. 6) that as σ_L increases the quantity (strength -σ_L) approaches zero following straight lines with $1/N$, but the slope gradually decreases, which suggests that the system size dependence of the strength gradually becomes weaker.

C. Avalanche size distribution

Due to faster crack growth, the LLS model shows different avalanche statistics than that of the ELS model. The numerical study of Hansen and Hemmer [9] suggests an apparent power law having exponent −4.5 in the avalanche distribution. Later, Kloster *et al.* [10] have shown analytically that for flat (uniform) distribution the LLS model does not have any power law asymptotics in avalanche statistics. We numerically study the avalanche distribution in the LLS model for different σ_L values (Fig. 7). We observe a similar deviation (lowering) of the distribution function for the smaller avalanche sizes as in the case of the ELS model. Also, the number of different avalanches gets reduced (the tail of the distribution disappears) with the increase of σ_L. This occurs due to damage localization [27], which ensures faster collapse of the bundle. In Fig. 7 we can see that for $\sigma_L=0.5$, avalanches of size 1 are the only possibility before total failure and their count is always less than 1, which clearly indicates the dominance of the instant failure situation.

IV. CONCLUSION

A lower cutoff in fiber threshold distribution excludes the presence of very weak fibers in a bundle. The weaker fibers mainly reduce the strength of a bundle. But, in practical purposes, we always try to build stronger and stronger materials (ropes, cables, etc.) from the fibrous elements. Therefore this situation (exclusion of weaker fibers) is very realistic. The failure dynamics of the ELS model almost remains unchanged in the presence of such lower cutoff (σ_L), whereas the avalanche size distributions show a systematic deviation (for small avalanches) from the mean field nature. At the limiting point ($\sigma_L \to 1/2$), we get a new power law (exponent −3/2) in avalanche distribution which can be explained from random walk statistics [12]. In the LLS model the avalanche statistics show a drastic change with the increase of σ_L. In both the models, the lower cutoff becomes bounded by an upper limit ($\sigma_L < 1/2$) beyond which the whole bundle

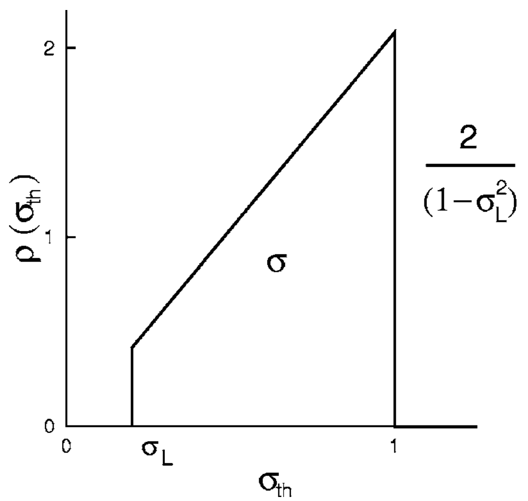

FIG. 8. The linearly increasing density of fiber strength having a lower cutoff (σ_L).

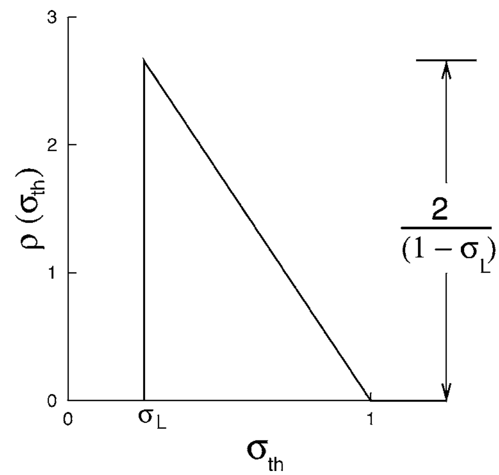

FIG. 9. The linearly decreasing density of fiber strength having a lower cutoff (σ_L).

fails at once, which has important consequences: It seems that the bundles show elastic-like response [17] up to $\sigma_L = 1/2$, above which they become perfectly brittle. We observe that the "weakest fiber breaking approach" [3,9] and the "equal load increment approach" [16,17] give similar results (in the ELS mode). In the equal load increment method, sometimes more than one fiber fails at the time of loading and this affects the whole failure dynamics, whereas the weakest fiber breaking approach ensures the single fiber (weakest among the intact fibers) failure at each step of loading. We consider the equal load increment method to be more practical from the experimental point of view. This approach helps to construct the recursion relations [16,17], which in turn show critical behavior [17,18] of the failure process. The instant failure situation is not limited to uniform threshold distribution, rather it is common in any type of distribution (see the Appendix). It seems that the instant failure represents the binary states of the bundle: intact (1 or high) and completely broken (0 or low) and here the bundle behaves like a classical switch in response to external load.

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APPENDIX

1. Case I: Linearly increasing density of fiber strength

We consider a bundle of fibers with linearly increasing density of strength (Fig. 8) having the normalized form

$$\rho(\sigma_{th}) = \frac{2\sigma_{th}}{1-\sigma_L^2}, \quad \sigma_L \leqslant \sigma_{th} \leqslant 1. \tag{A1}$$

We want to find out the bound of σ_L beyond which instant failure occurs in both ELS and LLS models.

a. ELS model

Following the weakest fiber breaking approach, the condition for "instant failure" is

$$N\int_{\sigma_L}^{\sigma_L[1+1/(N-1)]} \rho(\sigma_{th})d\sigma_{th} \geqslant 1, \tag{A2}$$

which gives

$$\sigma_L^2 \geqslant \frac{N-1}{3N-1}. \tag{A3}$$

Therefore, the bound (beyond which one-instant failure will occur) of the lower cutoff comes to be $\sigma_L < 1/\sqrt{3}$ for large N limit.

b. LLS model

The condition for instant failure for the LLS model is

$$n\int_{\sigma_L}^{\sigma_L[1+1/n]} \rho(\sigma_{th})d\sigma_{th} \geqslant 1, \tag{A4}$$

where n is the coordination number or the number of nearest neighbors. This gives

$$\sigma_L^2 \geqslant \frac{1}{\left(3+\frac{1}{n}\right)}. \tag{A5}$$

Now, as the dimension of the system increases, n goes towards infinity. Hence the above condition gives the bound of the lower cutoff as $\sigma_L < 1/\sqrt{3}$, which is identical to that in the equivalent ELS case.

2. Case II: Linearly decreasing density of fiber strength

Next we consider a bundle of fibers having linearly decreasing density of strength (Fig. 9) with the normalized form

$$\rho(\sigma_{th}) = \frac{2(1-\sigma_{th})}{(1-\sigma_L)^2}, \quad \sigma_L \leqslant \sigma_{th} \leqslant 1. \tag{A6}$$

a. ELS model

Following Eq. (A2), the condition for instant failure is

$$\sigma_L \geqslant \frac{N-1}{3N-1}, \tag{A7}$$

which sets the bound of the lower cutoff to $\sigma_L < 1/3$.

b. LLS model

Following Eq. (A4), the condition for instant failure is

$$\frac{2\sigma_L}{(1-\sigma_L)^2}\left(1-\sigma_L-\frac{\sigma_L}{2n}\right) \geqslant 1, \tag{A8}$$

which reduces to $2\sigma_L/(1-\sigma_L) \geqslant 1$ for $n \to \infty$ and sets the bound of the lower cutoff to $\sigma_L < 1/3$. Again this is identical to that in the ELS case.

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